Gauss-Bonnet Brane World Gravity with a Scalar Field

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Abstract. The effective four-dimensional, linearised gravity of a brane world model with one extra dimension and a single brane is analysed [1, 2]. The model includes higher order curvature terms (such as the Gauss-Bonnet term) and a conformally coupled scalar field. Large and small distance gravitational laws are derived. In contrast to the corresponding Einstein gravity models, it is possible to obtain solutions with localised gravity which are compatible with observations. Solutions with non-standard large distance Newtonian potentials are also described.

SECOND ORDER GRAVITY AND BRANE WORLDS

In the Randall-Sundrum II (RS) brane world scenario [3], we live on 3+1 dimensional brane embedded in a 4+1 dimensional bulk spacetime. As a result of the warping of the fifth dimension, the effective gravitational theory on the brane closely resembles that which is observed in our universe (except at very small distances). In this paper we will investigate an extended version of this scenario, which includes a conformally coupled bulk scalar field, ϕ, and an action which is of quadratic order in curvature. We will consider Z_2-symmetric solutions of the form

$$ds^2 = e^{2A(z)}(dx_4^2 + dz^2) \quad \text{with} \quad A = -\ln(1+|z|/\ell) \quad \text{and} \quad \phi/A = u = \text{constant} \tag{1}$$

which is the simplest generalisation of the RS model. The brane is located at $z = 0$. We will be mainly interested in solutions with $\ell > 0$ (which is required to localise gravity) and we take $u\ell > 0$ in order to avoid bulk singularities. We find that when the conformally coupled scalar field is added to the RS model, solutions with localised gravity can no longer be found. However, this problem can be fixed by including higher order gravity terms [4], as we will discuss in later sections.

In four dimensions, the gravitational field equations (for the vacuum) are taken to be $G_{ab} + \Lambda g_{ab} = 0$. These can be derived by looking for a rank 2 curvature tensor which (i) is symmetric, (ii) is divergence free, and (iii) depends only on the metric and its first two derivatives. In five dimensions the above conditions are satisfied by $G_{ab} + 2\alpha H_{ab} + \Lambda g_{ab} = 0$, where H_{ab} is the second order Lovelock tensor [5]. H_{ab} can be obtained from the variation of an action containing the Gauss-Bonnet term

$$\mathscr{L}_{\text{GB}} = R^{abcd}R_{abcd} - 4R_{ab}R^{ab} + R^2 \ . \tag{2}$$

Energy momentum is conserved in the corresponding gravitational theory and its vacuum is ghost-free (just as in Einstein gravity). Note that H_{ab} is the only quadratic curvature term which satisfies the above three conditions. In four dimensions its contribution to the field equations is trivial, and so it is usually ignored.

It is natural to expect the Gauss-Bonnet term to appear in the action of any five-dimensional theory, since even if it is not part of the fundamental theory, it is likely to be generated by quantum gravity corrections. A further reason for including the term in our brane model is that brane worlds are motivated by string theory, and the Gauss-Bonnet term also appears in low energy effective string actions.

Our model also includes a scalar field. In a string theory context this would correspond to the dilaton, or a moduli field coming from the compactification of other extra dimensions. As with the curvature terms, it is natural to include higher order scalar kinetic terms in the action. We will consider the general second order contribution

$$\mathscr{L}_2 = c_1 \mathscr{L}_{\mathrm{GB}} - 16 c_2 G_{ab} \nabla^a \phi \nabla^b \phi + 16 c_3 (\nabla\phi)^2 \nabla^2 \phi - 16 c_4 (\nabla\phi)^4 \; . \tag{3}$$

If ϕ is the dilaton, not all of the above coefficients are fixed by string theory. In fact low energy string theory actions suggest that R^2, $R_{ab}R^{ab}$, $R(\nabla\phi)^2$ and $R\nabla^2\phi$ are also possible. However we will not consider these since they all give ghosts at high energy (but they are not actually ruled out since we are dealing with a low energy effective action). If ϕ is a moduli field, the coefficients can be determined from the shape of the compactified dimensions.

We could also include third and higher order scalar kinetic terms, although for simplicity we will not consider them. In this case the full bulk action (in the string/Jordan frame) is

$$S_{\mathrm{Bulk}} = \frac{1}{2\kappa^2} \int d^5x \sqrt{-g} e^{-2\phi} \left\{ R - 4\omega (\nabla\phi)^2 + \mathscr{L}_2 - 2\Lambda \right\} \; . \tag{4}$$

Λ is the bulk cosmological constant.

The brane can be treated as a boundary of the bulk spacetime. In order for the Einstein-Hilbert action to consistent, we need to add the Gibbons-Hawking term [6] to the boundary action. Similarly, we will need a suitable boundary contribution to go with the Gauss-Bonnet term [7]

$$\mathscr{L}^{(b)}_{\mathrm{GB}} = \frac{4}{3}(3KK_{ac}K^{ac} - 2K_{ac}K^{cb}K^a{}_b - K^3) - 8G^{(4)}_{ab}K^{ab} \; . \tag{5}$$

Including the second order scalar field terms as well, the brane contribution to the action is

$$S_{\mathrm{brane}} = -\frac{1}{\kappa^2} \int d^4x \sqrt{-h} e^{-2\phi} \left\{ 2K + \mathscr{L}^{(b)}_2 + T \right\} \tag{6}$$

where T is the brane tension and

$$\mathscr{L}^{(b)}_2 = c_1 \mathscr{L}^{(b)}_{\mathrm{GB}} - 16 c_2 (K_{ab} - K h_{ab}) D^a \phi D^b \phi - 16 c_3 (n\cdot\nabla\phi) \left(\frac{1}{3}(n\cdot\nabla\phi)^2 + (D\phi)^2 \right) \; . \tag{7}$$

Variation of the action gives the generalised Israel junction conditions for the brane [1, 8]. These do not depend on the brane thickness (this is not true for other second order gravity terms).

LINEARISED BRANE GRAVITY

In general, a perturbation of the bulk metric will alter the position of the brane, and it is then necessary to change coordinates to put the brane back at $z=0$. We avoid this problem by working in a gauge in which the brane remains at $z=0$. This is achieved by considering a general perturbation of the bulk metric, and then using the components of the bulk field equations which are normal to the brane to determine $g_{\mu z}$ and g_{zz}. We obtain the perturbed metric [2]

$$ds^2 = e^{2A}\left[(\eta_{\mu\nu}+\gamma_{\mu\nu})dx^\mu dx^\nu + dz^2\right] - \ell e^A dz(dx^\mu\partial_\mu + dz\partial_z)\left(N_1(u)\psi + \frac{2}{u}\varphi\right) \ , \quad (8)$$

with

$$\gamma_{\mu\nu} = \bar{\gamma}_{\mu\nu} + 2(\zeta+\psi)\eta_{\mu\nu} - 2N_2(u)\frac{\partial_\mu\partial_\nu}{\Box_4}\psi \ , \quad (9)$$

where $\partial^\mu\bar{\gamma}_{\mu\nu}=0$ and $\eta^{\mu\nu}\bar{\gamma}_{\mu\nu}=0$. The perturbed scalar field is $\phi = uA+\varphi$. The fields $\psi \propto u\gamma - 8\varphi$ and ζ are linear combinations of φ and $\gamma = \eta^{\mu\nu}\gamma_{\mu\nu}$. The values of the various coefficients and the definitions of ψ and ζ can be found in ref. [2].

The bulk field equations give the wave equation for the transverse traceless part of the four-dimensional metric

$$\mu_\gamma(u)\left(\partial_z^2 - (3-2u)\ell^{-1}e^A\partial_z + f_\gamma^2(u)\Box_4\right)\bar{\gamma}_{\mu\nu} = 0 \ . \quad (10)$$

Note that there are no third or fourth order derivatives in the equation, despite the fact we have field equations which are of quadratic order in the curvature. The corresponding equation for the RS model has $u=0$ and $\mu_\gamma = f_\gamma^2 = 1$.

If either of μ_γ or f_γ^2 are negative, the kinetic term for $\bar{\gamma}_{\mu\nu}$ will have the wrong sign in the effective action, so the bulk will have graviton ghosts [9]. This requirement restricts the allowable ranges of the model's parameters.

The effective gravitational law on the brane is obtained from the junction conditions. For $\bar{\gamma}_{\mu\nu}$ we find

$$2\mu_\gamma\partial_z\bar{\gamma}_{\mu\nu} + m_\gamma^2\Box_4\bar{\gamma}_{\mu\nu} = -2\kappa^2\left\{S_{\mu\nu} - \frac{1}{3}\left(\eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box_4}\right)S\right\} \quad (11)$$

where $m_\gamma^2 = 8c_1(1-2u)/\ell$ and $S_{\mu\nu}$ is the perturbation of the brane energy momentum tensor. For the RS model we would just have $\partial_z\bar{\gamma}_{\mu\nu} = -\kappa^2\{S_{\mu\nu} - \cdots\}$.

If $m_\gamma^2 < 0$ then either the effective Planck mass on the brane is negative, or the vacuum has a non-trivial solution with spacelike momenta, i.e. the graviton spectrum includes a tachyon and the solution is unstable.

The graviton wave equation (10) is solved (for spacelike momenta) by

$$\bar{\gamma}_{\mu\nu} \propto e^{-ip\cdot x}e^{-(2-u)A}K_{2-u}\left(f_\gamma p\ell e^{-A}\right) \ . \quad (12)$$

Using a small p series expansion, we see (for $u<1$) that

$$\partial_z\bar{\gamma}_{\mu\nu} \approx -\frac{\ell f_\gamma^2 p^2}{2(1-u)}\bar{\gamma}_{\mu\nu} \quad (13)$$

for large distance scales ($1/p \gg \ell f_\gamma$). So we will have $\Box_4 \bar{\gamma}_{\mu\nu} \propto -\{S_{\mu\nu} - \cdots\}$, which is similar to the RS scenario, and will give a $1/r$ contribution to the Newton potential. The extra $\Box_4 \bar{\gamma}_{\mu\nu}$ term in the junction conditions gives a similar contribution at short distances (unlike the RS model), so the inclusion of higher gravity terms weakens the short distance gravity constraints on the model.

The behaviour of the scalar mode ψ is qualitatively similar. Its bulk field equation is the same as eq. (10), but with different parameters μ_ψ and f_ψ^2. Its junction condition is $2\mu_\psi \partial_z \psi + m_\psi^2 \Box_4 \psi = -\kappa^2 S$. As with the graviton modes, ghosts and tachyons are present for some parameter ranges.

The remaining degree of freedom, ζ, is pure gauge in the bulk, but its behaviour on the brane is given by $m_\zeta^2 \Box_4 \zeta = -\kappa^2 S$. It can be interpreted as the brane-bending mode.

EFFECTIVE FOUR DIMENSIONAL GRAVITY

Putting all the junction conditions together, we obtain the following expression for the induced metric perturbation

$$\gamma_{\mu\nu}(p) = 2\kappa^2 \left(D_\gamma(p) \left\{ S_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu} S \right\} + D_\psi(p)\eta_{\mu\nu} S + \eta_{\mu\nu} \frac{S}{m_\zeta^2 p^2} \right) \tag{14}$$

where

$$D_i(p) = \left\{ m_i^2 p^2 + 2\mu_i f_i p \frac{K_{u-1}(f_i \ell p)}{K_{2-u}(f_i \ell p)} \right\}^{-1} . \tag{15}$$

We have omitted the $p_\mu p_\nu / p^2$ dependent terms for simplicity. The three contributions to eq. (14) correspond to the graviton modes, the bulk scalar, and the ‘brane-bending’ mode. The Newton potential and the effective four-dimensional graviton propagator can be extracted from the above expression (14).

If $u < 1$, then the leading order behaviour of the perturbation (14) at large distances ($1/p \gg \ell f_{\gamma,\psi}$) is

$$\gamma_{\mu\nu}(p) \approx \frac{2}{M_{\rm Pl}^2 p^2} \left\{ S_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} S \right\} + \frac{1}{M_\Phi^2 p^2}\ \eta_{\mu\nu} S . \tag{16}$$

This describes Brans-Dicke gravity, with graviton mass $M_{\rm Pl} = \tilde{m}_\gamma/\kappa$ and scalar mass M_Φ given by

$$\frac{1}{M_\Phi^2} = 2\kappa^2 \left(\frac{1}{6\tilde{m}_\gamma^2} + \frac{1}{\tilde{m}_\psi^2} + \frac{1}{m_\zeta^2} \right) , \tag{17}$$

where $\tilde{m}_i^2 = \mu_i f_i^2/(1-u) + m_i^2$. To avoid conflict with solar system measurements we need $M_\Phi \gg M_{\rm Pl}$ [10]. This can be achieved for suitable parameter choices. If m_γ^2 is non-zero we also obtain Brans-Dicke gravity at shorter distances, but with different values of $M_{\rm Pl}$ and M_Φ.

As an example we will take $\omega = -1$ and $c_i = \alpha$ which would correspond to ϕ being the dilaton (with some extra symmetries).

For Einstein gravity ($\alpha = 0$), there is just one solution, with $u = \infty$. This is similar to a negative warp factor ($\ell < 0$) solution, and does not give localised gravity. Furthermore, when higher order terms are included, m_γ^2 is negative, and so the solution develops a tachyon. So we see that in this case the inclusion of higher gravity terms has not solved the solution's problems. In fact it has made them worse.

When the second order gravity terms are turned on ($\alpha > 0$), two extra solutions appear

$$\frac{\phi'}{A'} = u = \frac{3}{2} \pm \sqrt{\frac{3}{4} + \frac{\ell^2}{8\alpha}} \,. \tag{18}$$

The upper sign choice always gives a solution with instabilities, while the solution with the lower choice is stable if $u < 1/2$. We will just consider the latter solution for the remainder of this section.

Although the behaviour of the bulk graviton and scalar modes is qualitatively similar, the higher gravity terms give different contributions to the coefficients in the two wave equations. The degeneracy between scalar and graviton modes is broken by higher order gravity. In particular we can have

$$f_\gamma^2 = \frac{1-2u}{1-u} \ll f_\psi^2 = \frac{3}{3-2u} \tag{19}$$

for the above solution if $u \approx 1/2$.

When $1/p \gg \ell f_\psi$ (large distances), we find the couplings for the effective four-dimensional gravity are

$$M_{\mathrm{Pl}}^2 = 8M^3\alpha\ell^{-1}(2-u)f_\gamma^2 \tag{20}$$

$$M_\Phi^2 = 8M^3\alpha\ell^{-1}(3-2u) \tag{21}$$

It is therefore possible to obtain $M_\Phi \gg M_{\mathrm{Pl}}$ if the solution is fine-tuned to have $f_\gamma \ll 1$. This allows solar system constraints (from linearised gravity) to be satisfied. This is despite the fact that for the underlying five dimensional theory $M_{\mathrm{Pl}}^{(5)} \sim M_\Phi^{(5)}$. Note that the above fine-tuning of parameters is in addition to usual brane world fine-tuning of the cosmological constant, Λ, and the brane tension, T.

At intermediate ($\ell f_\gamma \ll 1/p \ll \ell f_\psi$) and short ($1/p \ll \ell f_\gamma$) distance scales, we find $M_\Phi^2 \leq 3M_{\mathrm{Pl}}^2$. However if these length scales are of geographical size, there will be no problem, since short distance constraints on scalar-tensor gravity are weak.

MODIFIED LARGE DISTANCE GRAVITY

So far we have assumed that the effects of the scalar field are smaller than the warping of space time (i.e $\phi'/A' = u < 1$). However if this is not true, it is possible to obtain solutions with non-standard Newton potentials at large distances.

If $1 < u < 2$ then the expression (13) is no longer valid, and instead we have

$$\partial_z\bar{\gamma}_{\mu\nu} \approx -\bar{\gamma}_{\mu\nu}\frac{2\,\Gamma(u-1)}{\ell\,\Gamma(2-u)}\left(\frac{p\ell}{2}\right)^{4-2u} \tag{22}$$

when $1/p \gg \ell f_\gamma$. Substituting eq. (22) into the junction condition (11), we find that $\bar{\gamma}_{\mu\nu}$ gives a non-standard $1/r^{2u-1}$ contribution to the large distance Newton potential. We now have only massive graviton bound states and no localised zero mode. However we can still obtain four-dimensional gravity at short distances from the $\Box_4$ terms in the junction conditions. The scalar modes have similar behaviour.

The resulting large and short distance gravity has some resemblance to the Dvali-Gabadadze-Porrati (DGP) model [11], especially for the special case of $u = 3/2$. We then have $1/r^2$ contributions to the large distance Newton law, as would normally occur in five-dimensional gravity.

Closer examination of the model reveals that it has more in common with quasilocalised brane gravity models [12] than the DGP model. Unfortunately this type of model either has ghosts or an unacceptably large scalar coupling [13]. By looking at the expression for the effective scalar mass (17) we see that $M_\Phi \gg M_{\rm Pl}$ is not possible unless m_ζ^2 (or m_ψ^2) is negative. Unfortunately if either of these parameters is negative the theory will have a ghost (or a tachyon if $m_\psi^2 < 0$ and $\mu_\psi > 0$).

The RS model also has $m_\zeta^2 < 0$, but this is not a problem. This model has two massless graviton zero modes, and an unphysical graviscalar zero mode. The brane-bending ghost zero mode cancels the graviscalar (a similar idea is used in the quantisation of QED). However this is not possible in quasilocalised models since there are no massless graviscalar or scalar states for the ghost to cancel with. For our model it is possible to cancel the contribution of the massive graviscalar modes by making ψ a ghost [2], although in contrast to the RS model, this is likely to be a physical ghost.

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